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LEFT INVARIANT POISSON STRUCTURES ON CLASSICAL NON-COMPACT SIMPLE LIE GROUPS

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ABSTRACT

A Poisson structure on a Lie group is called left invariant if the contravariant 2-tensor field π corresponding to the Poisson structure is left invariant. Explicit examples of such structures were known only for few cases, and in this paper, we give new examples of high rank left invariant Poisson structures for all non-compact classical real simple Lie groups. This result is equivalent to give constant solutions of the classical Yang–Baxter equation $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$ taking values in the space $\wedge^2\mathfrak{g}$ for these Lie groups.

Introduction

The main purpose of this paper is to give new examples of high rank left invariant Poisson structures on non-compact classical real simple Lie groups.

To state our results, we first explain some fundamental notions which we use in this paper. Let M be a differentiable manifold, and let $C^\infty(M)$ be the set of C^∞-functions on M. A Poisson structure on M is a skew-symmetric bilinear map $\{\ ,\ \} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying the following conditions:

$$\text{Jacobi rule}: \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0,$$

$$\text{Leibniz rule}: \{f, gh\} = \{f, g\}h + g\{f, h\}$$

$(f, g, h \in C^\infty(M))$. The Poisson structure $\{\ ,\ \}$ defines a contravariant 2-tensor field $\pi \in \Gamma(\wedge^2 TM)$ on M by

$$\langle \pi, df \wedge dg \rangle = \{f, g\},$$

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where $\langle\ ,\ \rangle$ is the natural pairing. Conversely, a 2-tensor field $\pi \in \Gamma(\wedge^2 TM)$ defines a bracket operation $\{\ ,\ \}$ by the above equality, and this $\{\ ,\ \}$ gives a Poisson structure on M if and only if π satisfies the equality $[\pi,\pi]_S = 0$, where $[\ ,\ ]_S\colon \Gamma(\wedge^2 TM) \times \Gamma(\wedge^2 TM) \to \Gamma(\wedge^3 TM)$ is the Schouten bracket defined by

$$[X\wedge Y, Z\wedge W]_S = [X,Z]\wedge Y\wedge W + X\wedge[Y,Z]\wedge W - [X,W]\wedge Y\wedge Z - X\wedge[Y,W]\wedge Z.$$

In this sense, we may say that π defines a Poisson structure on M if it satisfies $[\pi,\pi]_S = 0$. (For a general definition of the Schouten bracket and detailed explanation of Poisson manifolds, see [24] etc.)

Now, in this paper, we mainly consider the case M is a Lie group G. In this case, we say that a Poisson structure on G is left invariant if π is a left invariant 2-tensor field, i.e., π belongs to the space $\wedge^2\mathfrak{g}$ where $\mathfrak{g}$ is the Lie algebra of G, consisting of left invariant vector fields. In this situation, the equation $[\pi,\pi]_S = 0$ is called the "classical Yang–Baxter equation", which we abbreviate the CYB-equation (cf. [24; p. 173]). Clearly, from the above fact, the solution of the CYB-equation naturally corresponds to the left invariant Poisson structures on G, and finding solutions of this equation is the main subject of this paper.

Historically, the CYB-equation was first introduced by E. K. Sklyanin as the classical limit of the quantum YB-equation around in the 1980s, and reformulated in the form of the Schouten bracket by Gel'fand and Dorfman. Later, in the paper [2], Belavin and Drinfel'd classified the meromorphic solutions of the CYB-equation of the form

$$[X^{12}(u), X^{13}(u+v)] + [X^{12}(u), X^{23}(v)] + [X^{13}(u+v), X^{23}(v)] = 0$$

under some generic assumption, where the function $X(u)$ takes value in the tensor product $\mathfrak{g}\otimes\mathfrak{g}$ of a complex simple Lie algebra. (For the meaning of the above equation, see for example [2], [3].) The meromorphic solutions are extended to the whole complex plane and divided into three classes according as the rank of the lattice Γ consisting of the poles of $X(u)$. They completely classified the solutions in the cases rank $\Gamma = 2$ and 1. But in the remaining case rank $\Gamma = 0$, the classification was not done, and they only showed that $X(u)$ is equivalent to a rational function, and constructed some special examples. The problem we are considering now is just contained in this rank $\Gamma = 0$ case. Precisely, the solution of the CYB-equation in our sense coincides with the "constant" solution of the above equation

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$$

where r takes a value in the subspace $\wedge^2\mathfrak{g} \subset \mathfrak{g}\otimes\mathfrak{g}$, and $\mathfrak{g}$ is a real simple Lie algebra. (In their terminology, the triangle equations for constants. See [3;

p. 96] and §7 (4).) For this constant case, they gave examples of solutions for $\mathfrak{g} = \mathfrak{sl}(n, \mathbf{C})$ at the end of [2], but until now, solutions for other Lie algebras were not studied so well, as we will explain below.

In general, the CYB-equation $[\pi, \pi]_S = 0$ admits several types of degenerate solutions including the trivial solution $\pi = 0$ which corresponds to the Poisson structure defined by $\{f, g\} = 0$. The simplest non-trivial example is the case of solutions with rank $\pi = 2$, where rank π means the usual rank of a skew symmetric tensor of $\wedge^2\mathfrak{g}$. And in this case, there is a natural one-to-one correspondence between the set of solutions with rank $\pi = 2$ and the set of 2-dimensional subalgebras of $\mathfrak{g}$ (cf. Proposition 2). Hence, the complete classification of solutions of the CYB-equation including degenerate ones is almost "hopeless", as Belavin and Drinfel'd wrote in [2; p. 179]. Under these circumstances, as a first important problem, it is necessary to find generic (= high rank) solutions of the CYB-equation, and if possible, to classify such high rank solutions under the action of the adjoint groups.

Concerning this problem, we already know all solutions of the CYB-equation essentially for compact Lie groups (cf. Proposition 4), and from this result, it follows that the rank of generic solutions π (= max rank π) is equal to $2[1/2 \cdot \operatorname{rank} G]$. But for non-compact Lie groups, such an upper bound is not known yet in general, and it is the main purpose of this paper to present high rank solutions of the CYB-equation for all non-compact classical real simple Lie groups. As a by-product of this result, we can also construct a new class of Poisson–Lie group structures on G (cf. §7 (1)). Until now, such high rank examples of left invariant Poisson structures were known only for the groups $\mathrm{SL}(n, \mathbf{R}), \mathrm{SL}(n, \mathbf{C})$ and some other low dimensional (real or complex) Lie groups (cf. [2], [3], [6], [17], [19], [20], [21], [23], [25]). Our main results are summarized in the following theorem.

THEOREM 1: *Non-compact classical real simple Lie groups possess a left invariant Poisson structure π with the following rank:*

$$\mathrm{SL}(n, \mathbf{R})\text{: } \operatorname{rank}\pi = n(n-1),$$
$$\mathrm{SU}^*(2n)(n \geq 2)\text{: } \operatorname{rank}\pi = 4n(n-1),$$
$$\mathrm{Sp}(n, \mathbf{R})\text{: } \operatorname{rank}\pi = n(n+1),$$
$$\mathrm{SO}^*(2n)(n \geq 2, n \neq 4)\text{: } \operatorname{rank}\pi = n(n-1),$$
$$\mathrm{SU}(p,q)\text{: } \operatorname{rank}\pi = \begin{cases} 2pq & (p = q, q+1, q+2), \\ 2pq + 2[(p-q-1)/2] & (p \geq q+3), \end{cases}$$

$\mathrm{SO}(p,q)(p \geq q \geq 1)$: $\operatorname{rank}\pi =$

$$\begin{cases} pq + 2[(p-q)/4] & (q = \mathit{even}), \\ (p+1)(q-1) + 2[(3p-3q-1)/4] + c & (p \geq q+1, p = \mathit{even}, q = \mathit{odd}), \\ (p+1)(q-1) + 2[3(p-q)/4] + c & (p \geq q, p = \mathit{odd}, q = \mathit{odd}), \end{cases}$$

where

$$c = \begin{cases} 2 & p = q+1, q+2, q+4, q+5, \\ 4 & p = q = 3, \\ 0 & \mathit{otherwise}, \end{cases}$$

$$\mathrm{Sp}(p,q)(p \geq q \geq 1)\colon \operatorname{rank}\pi = \begin{cases} 2p^2 + 2p & (p = q), \\ 2pq + p & (p = \mathit{even} > q), \\ 2pq + p - 1 & (p = \mathit{odd} > q), \end{cases}$$

$\mathrm{SL}(n, \mathbf{C})^{\mathbf{R}}$: $\operatorname{rank}\pi = 2n(n-1)$,

$\mathrm{O}(2n+1, \mathbf{C})^{\mathbf{R}}$: $\operatorname{rank}\pi = 2n(n+1)$,

$\mathrm{Sp}(n, \mathbf{C})^{\mathbf{R}}$: $\operatorname{rank}\pi = 2n(n+1)$,

$$\mathrm{O}(2n, \mathbf{C})^{\mathbf{R}}\colon \operatorname{rank}\pi = \begin{cases} 2n^2 & (n = \mathit{even}), \\ 2(n^2-1) & (n = \mathit{odd}, n \neq 3), \\ 24 & (n = 3), \end{cases}$$

where $G^{\mathbf{R}}$ means the complex Lie group G itself, but considered as a real Lie group.

(The explicit form of each Poisson structure is stated in the proof of Theorem 1 given in §§2–6.) We can also show that among these solutions, the following cases give the highest rank solutions of the CYB-equation:

$$\begin{aligned} \mathrm{SL}(n, \mathbf{R})&\colon \max\operatorname{rank}\pi = n(n-1), \\ \mathrm{SU}(2,1)&\colon \max\operatorname{rank}\pi = 4, \\ \mathrm{SO}(3,1) \sim \mathrm{SL}(2, \mathbf{C})^{\mathbf{R}}&\colon \max\operatorname{rank}\pi = 4, \\ \mathrm{SO}(4,1) \sim \mathrm{Sp}(1,1)&\colon \max\operatorname{rank}\pi = 4, \\ \mathrm{SO}(3,2) \sim \mathrm{Sp}(2, \mathbf{R})&\colon \max\operatorname{rank}\pi = 6, \\ \mathrm{SO}(5,1) \sim \mathrm{SU}^*(4)&\colon \max\operatorname{rank}\pi = 8. \end{aligned}$$

(The symbol $\sim$ means the local isomorphism of Lie groups.) We show this fact by classifying high (even) dimensional subalgebras of each $\mathfrak{g}$, and check whether they admit a left invariant symplectic structure or not (cf. Proposition 2). For the remaining non-compact Lie groups, we do not know whether the values in Theorem 1 give the actual maximum of rank π or not at present. (See §7 (2), (3) and (5).)

Now, we explain the contents of this paper briefly. After reformulating our problem to the dual "symplectic" form in §1, we give a proof of Theorem 1 in

§§2–6. We prove this theorem using case by case construction because the construction depends heavily on the property of each Lie group. For example, among three Grassmann type Lie groups, $\mathrm{SU}(p,q)$ and $\mathrm{SO}(p,q)$ are essentially related to the structure of graded Lie algebras of the second kind, though $\mathrm{Sp}(p,q)$ is not. To construct such structures, not a little calculation on matrices is required for most cases. In the final section (§7), we give some results and comments related to left invariant Poisson structures, such as Poisson–Lie groups, the classification of solutions for $\mathrm{SL}(n,\mathbf{R})$, algebraic sets defined by the CYB-equation, etc.

1. Left invariant Poisson structures

In this section, we first review some known facts on left-invariant Poisson structures on G, which we use in this paper. To construct a high rank solution of $[\pi,\pi]_S = 0$, we reformulate this equation in the following dual form.

PROPOSITION 2 (cf. [2; p. 179], [7; p. 7], [16]): *There is a one-to-one correspondence between the set of solutions of the classical Yang–Baxter equation $[\pi,\pi]_S = 0$ and the set of pairs $(\mathfrak{g}',\omega)$, where $\mathfrak{g}'$ is an even dimensional subalgebra of $\mathfrak{g}$ and ω is a left invariant symplectic form on $\mathfrak{g}'$. In addition, under this correspondence, we have* $\operatorname{rank}\pi = \dim\mathfrak{g}'$.

(Here, a left invariant symplectic form on $\mathfrak{g}'$ actually implies a left invariant symplectic form on a Lie group whose Lie algebra is $\mathfrak{g}'$.)

The correspondence is given as follows (see [7; p. 7]): Let π be a solution of $[\pi,\pi]_S = 0$ with rank $\pi = 2k$, and we express it as

$$\pi = X_1 \wedge Y_1 + \cdots + X_k \wedge Y_k.$$

Then, from the condition $[\pi,\pi]_S = 0$, we know that $\mathfrak{g}' = \langle X_1,\ldots,X_k,Y_1,\ldots,Y_k\rangle$ is a $2k$-dimensional subalgebra of $\mathfrak{g}$, corresponding to π. We define a linear map $\hat{\pi}\colon \mathfrak{g}^* \to \mathfrak{g}$ by

$$\langle\hat{\pi}(\alpha),\beta\rangle = \pi(\alpha,\beta), \quad \alpha,\beta \in \mathfrak{g}^*.$$

Then, the symplectic form ω on $\mathfrak{g}'$ and $\pi \in \wedge^2\mathfrak{g}$ are related by

$$\omega(X,Y) = \pi(\hat{\pi}^{-1}(X),\hat{\pi}^{-1}(Y)), \quad X,Y \in \mathfrak{g}',$$
$$\pi(\alpha,\beta) = \omega((i^*(\alpha))^{\#},(i^*(\beta))^{\#}), \quad \alpha,\beta \in \mathfrak{g}^*,$$

where $i^*\colon \mathfrak{g}^* \to \mathfrak{g}'^*$ is the dual map of the inclusion $i\colon \mathfrak{g}' \to \mathfrak{g}$, and ${}^{\#}\colon \mathfrak{g}'^* \to \mathfrak{g}'$ is the map defined by $\omega(\alpha^{\#},X) = \alpha(X)$ for $\alpha \in \mathfrak{g}'^*, X \in \mathfrak{g}'$. It should be remarked that from this proposition, we know that the value "max rank π" is just equal to the maximum dimensional subalgebra of $\mathfrak{g}$ admitting a left invariant symplectic structure.

Two dimensional Lie algebras always admit a left invariant symplectic structure, and hence from the above proposition, there is a natural one-to-one correspondence between the set of two-dimensional subalgebras of $\mathfrak{g}$ and the set of solutions of the CYB-equation with rank $\pi = 2$. Of course, this fact also can be verified directly by considering the equality $[X \wedge Y, X \wedge Y]_S = 2X \wedge Y \wedge [X, Y] = 0$.

In a sense, a Poisson structure is a generalization of symplectic structures. In our situation, left invariant Poisson structure is of constant rank everywhere, and hence it defines a left invariant foliation on G whose leaves are all symplectic manifolds (cf. [21], [24]). The above proposition may be considered as a reformulation of this fact.

Concerning left invariant symplectic structures on Lie groups, the following results have already been proved by B. Y. Chu.

PROPOSITION 3 (cf. [9]): (1) *A left invariant symplectic structure does not exist on semi-simple Lie groups.*

(2) *Assume there exists a left invariant symplectic structure on a compact Lie group G. Then, G is abelian.*

(3) *Assume there exists a left invariant symplectic structure on a unimodular Lie group G. Then, G is solvable.*

In particular, from Proposition 2 and Proposition 3 (1), we know that $\max \operatorname{rank} \pi < \dim G$ for semi-simple Lie groups.

Concerning the solution of the CYB-equation on compact Lie groups, the following result is known. From this proposition, we may say that we essentially know all solutions of the CYB-equation for compact cases.

PROPOSITION 4 (cf. [7; p. 12]): *Assume G is compact, and let π be a solution of $[\pi, \pi]_S = 0$ on G. Then, π belongs to a space $\wedge^2 \mathfrak{a}$, where $\mathfrak{a}$ is an abelian subalgebra of $\mathfrak{g}$. In particular, we have* $\max \operatorname{rank} \pi = 2[1/2 \cdot \operatorname{rank} G]$ *for compact G.*

From this result, we immediately know that the solution of $[\pi, \pi]_S = 0$ for the group SO(3) (or SU(2)) must be trivial, i.e., $\pi = 0$ because rank $G = 1$. (See also [21].)

2. The case of $\mathrm{SU}(p, q)$

In the following, we give a proof of Theorem 1 using case by case construction of desired dimensional subalgebras $\mathfrak{g}'$ and symplectic forms ω on $\mathfrak{g}'$ (cf. Proposition

2). As for the Lie group $\mathrm{SL}(n,\mathbf{R})$, Belavin and Drinfel'd [2; p. 180] already constructed a solution with rank $\pi = n(n-1)$ (cf. §7 (2)), and hence we treat the remaining non-compact classical real simple Lie groups. In this section, we first consider the case $\mathrm{SU}(p,q)$ $(p \geq q \geq 1)$.

PROPOSITION 5: *There exists a left invariant Poisson structure π on the Lie group $\mathrm{SU}(p,q)$ $(p \geq q \geq 1)$ with*

$$\operatorname{rank}\pi = \begin{cases} 2pq & (p = q, q+1, q+2), \\ 2pq + 2[(p-q-1)/2] & (p \geq q+3). \end{cases}$$

Proof: We explicitly construct a desired dimensional subalgebra $\mathfrak{g}'$ of $\mathfrak{g} = \mathfrak{su}(p,q)$ and a symplectic form on $\mathfrak{g}'$. Before defining a subalgebra $\mathfrak{g}'$, we first prepare some notations. We denote by $M(p,q;\mathbf{K})$ the set of matrices of size (p,q) taking values in the field $\mathbf{K}$. For $A \in M(q,q;\mathbf{C})$, we define a new matrix $[A]$ of the same size by

$$[A]_{ij} = \begin{cases} a_{ij} & (i < j), \\ 0 & (i = j), \\ -a_{ij} & (i > j), \end{cases}$$

where a_{ij} is the (i,j)-component of A, and put

$$E = \begin{pmatrix} & & & & 1 \\ & 0 & & 1 & \\ & & \cdot & & \\ & 1 & & 0 & \\ 1 & & & & \end{pmatrix} \in M(q,q;\mathbf{R}).$$

Note that $[A]$ is skew-Hermitian if A is Hermitian. For $X \in M(q,q;\mathbf{C})$, the matrix XE is uniquely expressed as a sum $XE = A + B$, where ${}^t\overline{A} = A$ and ${}^t\overline{B} + B = 0$. In this situation, we define two matrices $X^{(1)}$ and $X^{(2)}$ by

$$X^{(1)} = [A] - B \quad \text{and} \quad X^{(2)} = E([A] + B)E.$$

Then, we have the following lemma.

LEMMA 6: *Assume $X, Y \in M(q,q;\mathbf{C})$ and $P, Q \in M(p-q,q;\mathbf{C})$.*

(1) *$X^{(1)}$, $X^{(2)} \in \mathfrak{u}(q)$ and $\operatorname{Tr} X^{(1)} + \operatorname{Tr} X^{(2)} = 0$.*

(2) *$X^{(2)} = EX^{(1)}E + EX - {}^t\overline{X}E$.*

(3) *For $Z = ({}^t\overline{P}Q - {}^t\overline{Q}P)E$, we have*

$$Z^{(1)} = {}^t\overline{Q}P - {}^t\overline{P}Q, \quad Z^{(2)} = E({}^t\overline{P}Q - {}^t\overline{Q}P)E.$$

(4) *For* $Z = X^{(1)}Y - Y^{(1)}X + XY^{(2)} - YX^{(2)}$, *we have*

$$\begin{aligned} Z^{(1)} &= [X^{(1)}, Y^{(1)}] + X{}^t\overline{Y} - Y{}^t\overline{X}, \\ Z^{(2)} &= [X^{(2)}, Y^{(2)}] + {}^t\overline{X}Y - {}^t\overline{Y}X. \end{aligned}$$

Proof: The properties (1) and (2) follow immediately from the definition, and the property (3) follows from the fact ${}^t\overline{P}Q - {}^t\overline{Q}P \in \mathfrak{u}(q)$. Now, we prove (4). We express the matrix YE as a sum $YE = C + D$ (${}^t\overline{C} = C$ and ${}^t\overline{D} + D = 0$). Then, we have the decomposition

$$\begin{gathered} (X^{(1)}Y - Y^{(1)}X + XY^{(2)} - YX^{(2)})E = \\ ([A]C - [C]A + A[C] - C[A]) \\ +([A]D - D[A] + AD + DA + B[C] - [C]B - BC - CB). \end{gathered}$$

In the last expression, the first term is Hermitian and the second term is skew-Hermitian, and hence we have

$$\begin{aligned} Z^{(1)} =& [[A]C - [C]A + A[C] - C[A]] \\ & - [A]D + D[A] - AD - DA - B[C] + [C]B + BC + CB. \end{aligned}$$

On the other hand, we can easily prove the equality

$$[[A]C + A[C]] + \begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{qq} \end{pmatrix} \begin{pmatrix} c_{11} & & 0 \\ & \ddots & \\ 0 & & c_{qq} \end{pmatrix} = [A][C] + AC,$$

where a_{ij} (resp. c_{ij}) is the (i,j)-component of A (resp. C). In fact, the (i,j)-components of both sides are equal to

$$\begin{array}{ll} a_{ii}c_{ij} + 2\sum_{i<k<j} a_{ik}c_{kj} + a_{ij}c_{jj} & (i < j), \\ a_{ii}c_{ii} & (i = j), \\ a_{ij}c_{jj} + 2\sum_{j<k<i} a_{ik}c_{kj} + a_{ii}c_{ij} & (i > j). \end{array}$$

By using this equality, we have

$$\begin{aligned} Z^{(1)} =& [A][C] - [C][A] + AC - CA \\ & - [A]D + D[A] - AD - DA - B[C] + [C]B + BC + CB, \end{aligned}$$

which is equal to $[X^{(1)}, Y^{(1)}] + X{}^t\overline{Y} - Y{}^t\overline{X}$. The second equality of (4) can be proved directly from (2) and the first one. ∎

Now, we define a subalgebra $\mathfrak{g}'$ of $\mathfrak{su}(p,q)$ as follows. We express an element of $\mathfrak{su}(p,q)$ in the block form

$$\begin{array}{cc} \begin{pmatrix} X & -{}^t\overline{Y} & {}^t\overline{P} \\ Y & Z & {}^t\overline{Q} \\ P & Q & W \end{pmatrix} & \begin{array}{c} q \\ p-q \\ q \end{array} \\ \begin{array}{ccc} q & p-q & q \end{array} & \end{array} \qquad X,\, W \in \mathfrak{u}(q),\ Z \in \mathfrak{u}(p-q),$$

where $\operatorname{Tr} X + \operatorname{Tr} Z + \operatorname{Tr} W = 0$. We put

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} X^{(1)} & 0 & X \\ 0 & 0 & 0 \\ {}^t\overline{X} & 0 & X^{(2)} \end{pmatrix} \middle| X \in M(q,q;\mathbf{C}) \right\},$$

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 0 & {}^t\overline{P} & 0 \\ -P & 0 & PE \\ 0 & E{}^t\overline{P} & 0 \end{pmatrix} \middle| P \in M(p-q,q;\mathbf{C}) \right\},$$

and let $\mathfrak{g}_3$ be a $2[(p-q-1)/2]$-dimensional subalgebra of the space

$$\left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{pmatrix} \middle| K = \begin{pmatrix} a_1 i & & 0 \\ & \ddots & \\ 0 & & a_{p-q} i \end{pmatrix},\ \begin{array}{c} a_k \in \mathbf{R} \\ \operatorname{Tr} K = 0 \end{array} \right\}.$$

(In the case $p \le q+2$, we put $\mathfrak{g}_3 = \{0\}$.) Using Lemma 6, we can easily check that $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$ is a subalgebra of $\mathfrak{su}(p,q)$, satisfying the following bracket table:

[,]	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
$\mathfrak{g}_1$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	0
$\mathfrak{g}_2$		$\mathfrak{g}_1$	$\mathfrak{g}_2$
$\mathfrak{g}_3$			0 .

For example, the properties $[\mathfrak{g}_1,\mathfrak{g}_1] \subset \mathfrak{g}_1$, $[\mathfrak{g}_1,\mathfrak{g}_2] \subset \mathfrak{g}_2$, $[\mathfrak{g}_2,\mathfrak{g}_2] \subset \mathfrak{g}_1$ follow from (4), (2), (3) in Lemma 6, respectively. The dimensions of $\mathfrak{g}_1$, $\mathfrak{g}_2$ are equal to $2q^2$ and $2q(p-q)$, and hence, $\mathfrak{g}'$ is the desired dimensional subalgebra.

Finally, we define a symplectic form ω on $\mathfrak{g}'$ as follows. First, we define $\alpha \in \mathfrak{g}'^*$ by

$$\alpha\left(\begin{pmatrix} X^{(1)} & {}^t\overline{P} & X \\ -P & K & PE \\ {}^t\overline{X} & E{}^t\overline{P} & X^{(2)} \end{pmatrix} \right) = i \cdot \operatorname{Tr}(XE - E{}^t\overline{X})/2 \in \mathbf{R}.$$

Then, we can prove that the exact 2-form $-d\alpha$ is non-degenerate on the subspace $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, and $-d\alpha(\mathfrak{g}_3, \mathfrak{g}') = 0$. For example, by expressing the matrix

$$\begin{pmatrix} X^{(1)} & 0 & X \\ 0 & 0 & 0 \\ {}^t\overline{X} & 0 & X^{(2)} \end{pmatrix} \in \mathfrak{g}_1$$

simply as X etc., we have $-d\alpha(X,Y) = \alpha([X,Y]) = i \cdot \mathrm{Tr}\,(ZE - E^t\overline{Z})/2$, where $Z = X^{(1)}Y - Y^{(1)}X + XY^{(2)} - YX^{(2)}$. Since $(ZE - E^t\overline{Z})/2$ is the skew-Hermitian part of ZE, the value $\alpha([X,Y])$ is equal to

$$i \cdot \mathrm{Tr}([A]D - D[A] + AD + DA + B[C] - [C]B - BC - CB),$$

where $XE = A + B$, $YE = C + D$ (${}^t\overline{A} = A, {}^t\overline{C} = C, {}^t\overline{B} + B = {}^t\overline{D} + D = 0$. Remember the proof of Lemma 6 (4).) Then, we immediately know that this is equal to $2i \cdot \mathrm{Tr}\,(AD - BC)$, and the non-degeneracy of $-d\alpha$ on $\mathfrak{g}_1$ follows from this expression. The non-degeneracy of $-d\alpha$ on $\mathfrak{g}_2$ and the properties $-d\alpha(\mathfrak{g}_1, \mathfrak{g}_2) = -d\alpha(\mathfrak{g}_3, \mathfrak{g}') = 0$ can be proved easily. Next, we fix a symplectic form ω' on the abelian subalgebra $\mathfrak{g}_3$. By using the decomposition $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$, we can naturally extend ω' to the 2-form on $\mathfrak{g}'$, which is also closed on account of the property $[\mathfrak{g}', \mathfrak{g}'] \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Then, by putting $\omega = -d\alpha + \omega'$, we obtain the desired symplectic form on $\mathfrak{g}'$. ∎

Remark: The above construction is essentially related to the graded Lie algebra of the second kind. The Lie algebra $\mathfrak{su}(p,q)$ possesses a graded Lie algebra structure $\mathfrak{su}(p,q) = \mathfrak{l}_{-2} \oplus \cdots \oplus \mathfrak{l}_2$ satisfying $\dim \mathfrak{l}_{\pm 2} = 1$. (For the explicit decomposition, see [8], [14].) We denote by E the element of $\mathfrak{l}_0$ giving the gradation, i.e., $\mathfrak{l}_p \doteq \{X \in \mathfrak{su}(p,q) \mid [E,X] = pX\}$ for $p = -2 \sim 2$. Then the $2(p+q-1)$-dimensional subalgebra $\langle E \rangle \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2$ admits a left invariant symplectic form ω defined by $\omega(X,Y) = \mathfrak{l}_2$-component of $[X,Y]$. And we can easily check that this subalgebra has a trivial intersection with the subalgebra $\mathfrak{su}(p-1,q-1)$ obtained by deleting the outer layer of $\mathfrak{su}(p,q)$. Next, we once again construct a similar subalgebra starting from $\mathfrak{su}(p-1,q-1)$, and repeat this procedure q times. Then, collecting these subalgebras and symplectic forms, we finally obtain the subalgebra $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ and the left invariant symplectic form $-d\alpha$ on it constructed in the above proof. The final subalgebra $\mathfrak{g}'$ is obtained by adding the abelian Lie algebra $\mathfrak{g}_3$ which is contained in the remaining core compact subalgebra $\mathfrak{su}(p-q,0) = \mathfrak{su}(p-q) \subset \mathfrak{su}(p,q)$.

3. The case of $\mathrm{SO}(p,q)$

In the case of $G = \mathrm{SO}(p,q)$ $(p \geq q \geq 1)$, there exists a Poisson structure with $\mathrm{rank}\,\pi \doteqdot pq + (p-q)/2$. The construction is almost the same as $\mathrm{SU}(p,q)$. But, it is a little more complicated and we must divide the construction into several cases according to the parity of p and q. Precisely, we have the following proposition.

PROPOSITION 7: *There exists a left invariant Poisson structure* π *on the Lie group* $\mathrm{SO}(p,q)$ $(p \geq q \geq 1)$ *with*

$$\operatorname{rank}\pi = \begin{cases} pq + 2[(p-q)/4] & (q = \mathit{even}), \\ (p+1)(q-1) + 2[(3p-3q-1)/4] + c & (p \geq q+1,\ p = \mathit{even},\ q = \mathit{odd}), \\ (p+1)(q-1) + 2[3(p-q)/4] + c & (p \geq q,\ p = \mathit{odd},\ q = \mathit{odd}), \end{cases}$$

where

$$c = \begin{cases} 2 & p = q+1,\ q+2,\ q+4,\ q+5, \\ 4 & p = q = 3, \\ 0 & \text{otherwise.} \end{cases}$$

Proof: We construct a subalgebra $\mathfrak{g}'$ of $\mathfrak{o}(p,q)$ with the desired dimension by dividing roughly into two cases according as the parity of q.

(i) The case $q =$ even.

We put $q = 2r$. In this case, we can construct a subalgebra $\mathfrak{g}'$ almost in the same way as $\mathfrak{g} = \mathfrak{su}(p,q)$. First, as in §2, we put

$$[A]_{ij} = \begin{cases} a_{ij} & (i < j), \\ 0 & (i = j), \\ -a_{ij} & (i > j), \end{cases}$$

for $A \in M(q,q;\mathbf{R})$. In this case, $[A]$ is skew-symmetric if A is symmetric. In addition, we put

$$E = \begin{pmatrix} & & & I_2 \\ 0 & & I_2 & \\ & \cdot & & \\ & \cdot & & \\ & I_2 & & 0 \\ I_2 & & & \end{pmatrix} \in M(2r,2r;\mathbf{R}), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

For $X \in M(2r,2r;\mathbf{R})$, the matrix XE is uniquely expressed as a sum $XE = A + B$, where ${}^tA = A$ and ${}^tB + B = 0$. In this situation, we put

$$X^{(1)} = [A] - B \quad \text{and} \quad X^{(2)} = E([A] + B)E$$

as before. Then, we have the following lemma. This lemma can be proved in the same way as Lemma 6, and we omit the proof.

LEMMA 8: *Assume* X, $Y \in M(2r,2r;\mathbf{R})$ *and* P, $Q \in M(p-2r,2r;\mathbf{R})$.

(1) $X^{(1)}$, $X^{(2)} \in \mathfrak{o}(2r)$.

(2) $X^{(2)} = EX^{(1)}E + EX - {}^tXE$.

(3) *For* $Z = ({}^tPQ - {}^tQP)E$, *we have*

$$Z^{(1)} = {}^tQP - {}^tPQ, \quad Z^{(2)} = E({}^tPQ - {}^tQP)E.$$

(4) *For* $Z = X^{(1)}Y - Y^{(1)}X + XY^{(2)} - YX^{(2)}$, *we have*

$$\begin{aligned} Z^{(1)} &= [X^{(1)}, Y^{(1)}] + X{}^tY - Y{}^tX, \\ Z^{(2)} &= [X^{(2)}, Y^{(2)}] + {}^tXY - {}^tYX. \end{aligned}$$

Now, under these preliminaries, we put

$$\mathfrak{g}_1 = \left\{ \begin{array}{l} \left(\begin{array}{ccc} X^{(1)} & 0 & X \\ 0 & 0 & 0 \\ {}^tX & 0 & X^{(2)} \end{array} \right) \begin{array}{l} {\scriptstyle q} \\ {\scriptstyle p-q} \\ {\scriptstyle q} \end{array} \\ \quad \begin{array}{ccc} {\scriptstyle q} & {\scriptstyle p-q} & {\scriptstyle q} \end{array} \end{array} \middle| \, X \in M(q,q;\mathbf{R}) \right\},$$

$$\mathfrak{g}_2 = \left\{ \begin{array}{l} \left(\begin{array}{ccc} 0 & {}^tP & 0 \\ -P & 0 & PE \\ 0 & E{}^tP & 0 \end{array} \right) \begin{array}{l} {\scriptstyle q} \\ {\scriptstyle p-q} \\ {\scriptstyle q} \end{array} \\ \quad \begin{array}{ccc} {\scriptstyle q} & {\scriptstyle p-q} & {\scriptstyle q} \end{array} \end{array} \middle| \, P \in M(p-q,q;\mathbf{R}) \right\},$$

and let $\mathfrak{g}_3$ be a $2[(p-q)/4]$-dimensional abelian subalgebra of the space

$$\left\{ \left(\begin{array}{ccc} 0 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 0 \end{array} \right) \middle| \, K \in \mathfrak{o}(p-q) \right\}.$$

Then, by using the properties in Lemma 8, we can easily show that the bracket satisfies the following table:

[,]	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$
$\mathfrak{g}_1$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	0
$\mathfrak{g}_2$		$\mathfrak{g}_1$	$\mathfrak{g}_2$
$\mathfrak{g}_3$			0 .

Hence, by putting $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$, $\mathfrak{g}'$ is a subalgebra of $\mathfrak{o}(p,q)$ with the desired dimension.

Next, we define a left invariant symplectic form on $\mathfrak{g}'$. For this purpose, we first define the element $\alpha \in \mathfrak{g}'^*$ by

$$\alpha \left(\left(\begin{array}{ccc} X^{(1)} & {}^tP & X \\ -P & K & PE \\ {}^tX & E{}^tP & X^{(2)} \end{array} \right) \right) = x^{(1)}_{12} + x^{(1)}_{34} + \cdots + x^{(1)}_{2r-1,2r},$$

where $x_{ij}^{(1)}$ is the (i,j)-component of $X^{(1)} \in M(2r, 2r; \mathbf{R})$. Then, $-d\alpha$ is non-degenerate on the subspace $\mathfrak{g}_1 \oplus \mathfrak{g}_2$, and $-d\alpha(\mathfrak{g}_3, \mathfrak{g}') = 0$. For example, by expressing the matrix

$$\begin{pmatrix} X^{(1)} & 0 & X \\ 0 & 0 & 0 \\ {}^tX & 0 & X^{(2)} \end{pmatrix} \in \mathfrak{g}_1$$

simply as X etc., we have $-d\alpha(X, Y) = \alpha([X, Y]) = z_{12}^{(1)} + \cdots + z_{2r-1,2r}^{(1)}$, where $z_{ij}^{(1)}$ is the (i,j)-component of $Z^{(1)} = [X^{(1)}, Y^{(1)}] + X^tY - Y^tX$. We express the matrices X and Y in terms of (2,2)-blocks as follows:

$$X = \begin{pmatrix} X_{11} & \cdots & X_{1r} \\ & \cdots & \\ X_{r1} & \cdots & X_{rr} \end{pmatrix}, \quad Y = \begin{pmatrix} Y_{11} & \cdots & Y_{1r} \\ & \cdots & \\ Y_{r1} & \cdots & Y_{rr} \end{pmatrix},$$

where X_{ij}, $Y_{ij} \in M(2, 2; \mathbf{R})$. Then, the matrix $X^{(1)}$ can be expressed in the following block form:

$$X_{ij}^{(1)} = \begin{cases} {}^tX_{j,r+1-i} & (i < j), \\ X_{i,r+1-i}^0 & (i = j), \\ -X_{i,r+1-j} & (i > j), \end{cases}$$

where we put $P^0 = \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$ for the matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Using this expression, we can show that $z_{12}^{(1)} + \cdots + z_{2r-1,2r}^{(1)}$ is equal to the $(1,2)$-component of the $(2,2)$-matrix

$$\sum_{i,j} (X_{ij}^{(1)} Y_{ji}^{(1)} - Y_{ij}^{(1)} X_{ji}^{(1)} + X_{ij}{}^tY_{ij} - Y_{ij}{}^tX_{ij})$$
$$= \sum_{i+j \leq r+1} (X_{ij}{}^tY_{ij} - Y_{ij}{}^tX_{ij}) - \sum_{r+2 \leq i+j} ({}^tX_{ij}Y_{ij} - {}^tY_{ij}X_{ij}),$$

and the non-degeneracy of $-d\alpha$ on $\mathfrak{g}_1$ follows immediately from this formula. Other properties on $-d\alpha$ can be proved in the same way. Next, let ω' be a left invariant symplectic form on $\mathfrak{g}_3$. We can naturally extend ω' to the form on $\mathfrak{g}'$ by using the decomposition $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \mathfrak{g}_3$. Then, since $[\mathfrak{g}', \mathfrak{g}'] \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$, we can easily check that ω' is also closed on $\mathfrak{g}'$. Hence, by putting $\omega = -d\alpha + \omega'$, we obtain the desired symplectic form on $\mathfrak{g}'$.

(ii) The case $q =$ odd.

In this case, we first prove the following lemma, corresponding to the case $q = 1$ in Proposition 7. We later use this lemma in the proof of Proposition 7 for general cases $q = \text{odd} \geq 3$.

LEMMA 9: *There exists a left invariant Poisson structure* π *on the Lie group* $\mathrm{SO}(s,1)$ $(s \geq 1)$ *with*

$$\operatorname{rank}\pi = \begin{cases} 2[3s/4] - 2 + c & (s = \text{even}), \\ 2[3(s-1)/4] + c & (s = \text{odd}), \end{cases}$$

where

$$c = \begin{cases} 2 & s = 2,\ 3,\ 5,\ 6, \\ 0 & \text{otherwise}. \end{cases}$$

Proof: The case $s = 1$ is trivial since $\mathfrak{o}(1,1)$ is the 1-dimensional abelian Lie algebra. In the case $s = 2$, $\mathfrak{o}(2,1)$ is isomorphic to $\mathfrak{sl}(2,\mathbf{R})$, and we already know that it admits a left invariant Poisson structure with $\operatorname{rank}\pi = 2$ (cf. [2; p. 180], §7 (2)).

Next, we construct a Poisson structure on $\mathfrak{o}(3,1)$ with $\operatorname{rank}\pi = 4$. (See also [25; p. 22].) We denote by E_{ij} the matrix such that the (i,j)-component is 1 and other components are all zero, and put

$$\begin{aligned} X_1 &= E_{12} - E_{21} + E_{24} + E_{42}, & X_2 &= E_{13} - E_{31} + E_{34} + E_{43}, \\ X_3 &= E_{14} + E_{41}, & X_4 &= E_{23} - E_{32}. \end{aligned}$$

Then, it is easy to check that $\mathfrak{g}' = \langle X_1, \ldots, X_4 \rangle$ is a 4-dimensional subalgebra of $\mathfrak{o}(3,1)$. We define $\alpha \in \mathfrak{g}'^*$ by

$$\alpha(X) = X_1\text{-component of } X$$

for $X \in \mathfrak{g}'$. Then, the exact 2-form $\omega = -d\alpha$ gives the desired symplectic form on $\mathfrak{g}'$.

In the case of $s = 5$, 6, we put

$$\begin{aligned} &X_i = E_{1i} - E_{i1} + E_{i,s+1} + E_{s+1,i} \ (2 \leq i \leq 5), \\ &Y_1 = E_{23} - E_{32} + E_{45} - E_{54}, \quad Y_2 = E_{24} - E_{42} - E_{35} + E_{53}, \\ &Y_3 = E_{25} - E_{52} + E_{34} - E_{43}, \quad Y_4 = E_{1,s+1} + E_{s+1,1}. \end{aligned}$$

Then $\mathfrak{g}' = \langle X_2, \ldots, X_5, Y_1, \ldots, Y_4 \rangle$ is the 8-dimensional subalgebra of $\mathfrak{o}(s,1)$. Let α be the element of $\mathfrak{g}'^*$ defined by

$$\alpha(X) = X_2\text{-component of } X$$

for $X \in \mathfrak{g}'$. Then, we can directly check that $\omega = -d\alpha$ is the symplectic form on $\mathfrak{g}'$.

Next, we consider the remaining general cases $s = 4$ and $s \geq 7$. Remark that $c = 0$ for these cases. We divide the proof into three types according to the value of s. We first put

$$X_i = E_{1i} - E_{i1} + E_{i,s+1} + E_{s+1,i},$$
$$Y_j = E_{2j,2j+1} - E_{2j+1,2j},$$

for $2 \leq i \leq s$ and $1 \leq j \leq [(s-1)/2]$. Then, the bracket operations of these matrices are given by

$$[X_{2j}, Y_j] = X_{2j+1}, \quad [X_{2j+1}, Y_j] = -X_{2j}, \quad 1 \leq j \leq [(s-1)/2],$$

and other brackets are all zero. In particular, $\langle X_i, Y_j \rangle$ is a $[3(s-1)/2]$-dimensional subalgebra of $\mathfrak{o}(s,1)$.

(1) The case $s = 4k+1$, $4k+2$.
We put $\mathfrak{g}' = \langle X_2, \ldots, X_{4k+1}, Y_1, \ldots, Y_{2k} \rangle$. Then, $\mathfrak{g}'$ is a $6k$-dimensional subalgebra of $\mathfrak{o}(s,1)$. We denote by $\alpha_i, \beta_j \in \mathfrak{g}'^*$ the dual basis of $\mathfrak{g}' = \langle X_i, Y_j \rangle$. Then, from the above bracket operations, we have

$$d\alpha_{2j+1} = -\alpha_{2j} \wedge \beta_j, \quad d\alpha_{2j} = \alpha_{2j+1} \wedge \beta_j, \quad d\beta_j = 0$$

for $1 \leq j \leq 2k$. Hence, the 2-form

$$\omega = \sum_{j=1}^{2k} \alpha_{2j} \wedge \alpha_{2j+1} + \sum_{i=1}^{k} \beta_{2i-1} \wedge \beta_{2i}$$

gives the desired symplectic form on $\mathfrak{g}'$.

(2) The case $s = 4k-1$.
We put $\mathfrak{g}' = \langle X_2, \ldots, X_{4k-1}, Y_1, \ldots, Y_{2k-2} \rangle$. Then $\mathfrak{g}'$ is a $(6k-4)$-dimensional subalgebra of $\mathfrak{o}(4k-1,1)$. We denote by α_i, $\beta_j \in \mathfrak{g}'^*$ the dual basis of $\langle X_i, Y_j \rangle$, as above. Then, we have

$$d\alpha_{2j+1} = -\alpha_{2j} \wedge \beta_j, \quad d\alpha_{2j} = \alpha_{2j+1} \wedge \beta_j, \quad d\beta_j = 0,$$

for $1 \leq j \leq 2k-2$, and $d\alpha_{4k-2} = d\alpha_{4k-1} = 0$. Hence, the 2-form

$$\omega = \sum_{j=1}^{2k-1} \alpha_{2j} \wedge \alpha_{2j+1} + \sum_{i=1}^{k-1} \beta_{2i-1} \wedge \beta_{2i}$$

gives the desired symplectic form on $\mathfrak{g}'$.

(3) The case $s = 4k$.
We put $\mathfrak{g}' = \langle X_2, \ldots, X_{4k}, Y_1, \ldots, Y_{2k-1} \rangle$. Then $\mathfrak{g}'$ is a $(6k-2)$-dimensional subalgebra of $\mathfrak{o}(4k,1)$, and by using a dual basis as above, we have

$$d\alpha_{2j+1} = -\alpha_{2j} \wedge \beta_j, \quad d\alpha_{2j} = \alpha_{2j+1} \wedge \beta_j, \quad d\beta_j = 0$$

for $1 \leq j \leq 2k-1$, and $d\alpha_{4k} = 0$. Hence, the 2-form

$$\omega = \sum_{j=1}^{2k-1} \alpha_{2j} \wedge \alpha_{2j+1} + \sum_{i=1}^{k-1} \beta_{2i-1} \wedge \beta_{2i} + \alpha_{4k} \wedge \beta_{2k-1}$$

gives the desired symplectic form on $\mathfrak{g}'$. ∎

Next, we give the proof of Proposition 7 for the case $q = \text{odd} \geq 3$. We put $q = 2r+1$ $(r \geq 1)$. Using the same notations as in the case of $q = $ even, we put

$$\mathfrak{g}_1 = \left\{ \begin{array}{cl} \left(\begin{array}{cccc} X^{(1)} & 0 & 0 & X \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ {}^tX & 0 & 0 & X^{(2)} \end{array} \right) & \begin{array}{c} q-1 \\ p-q+1 \\ 1 \\ q-1 \end{array} \\ \begin{array}{cccc} q-1 & p-q+1 & 1 & q-1 \end{array} & \end{array} \middle| \; X \in M(q-1, q-1; \mathbf{R}) \right\},$$

$$\mathfrak{g}_2 = \left\{ \begin{array}{cl} \left(\begin{array}{cccc} 0 & {}^tP & 0 & 0 \\ -P & 0 & 0 & PE \\ 0 & 0 & 0 & 0 \\ 0 & E^tP & 0 & 0 \end{array} \right) & \begin{array}{c} q-1 \\ p-q+1 \\ 1 \\ q-1 \end{array} \\ \begin{array}{cccc} q-1 & p-q+1 & 1 & q-1 \end{array} & \end{array} \middle| \; P \in M(p-q+1, q-1; \mathbf{R}) \right\},$$

$$\mathfrak{g}_3 = \left\{ \begin{array}{cl} \left(\begin{array}{cccc} 0 & 0 & {}^tQ & 0 \\ 0 & 0 & 0 & 0 \\ Q & 0 & 0 & -QE \\ 0 & 0 & E^tQ & 0 \end{array} \right) & \begin{array}{c} q-1 \\ p-q+1 \\ 1 \\ q-1 \end{array} \\ \begin{array}{cccc} q-1 & p-q+1 & 1 & q-1 \end{array} & \end{array} \middle| \; Q \in M(1, q-1; \mathbf{R}) \right\},$$

where E is the matrix defined by

$$E = \begin{pmatrix} & & & I_2 \\ & 0 & I_2 & \\ & & \cdot & \\ & \cdot & & \\ & I_2 & 0 & \\ I_2 & & & \end{pmatrix} \in M(2r, 2r; \mathbf{R}), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

as before. In addition, we imbed the subalgebra of $\mathfrak{o}(s,1)$ $(s = p-q+1)$ constructed in Lemma 9 as a subspace of the following space in a natural way:

$$\left\{ \begin{array}{c} \left(\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & K & L & 0 \\ 0 & {}^tL & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \begin{array}{c} q-1 \\ p-q+1 \\ 1 \\ q-1 \end{array} \\ \begin{array}{cccc} q-1 & p-q+1 & 1 & q-1 \end{array} \end{array} \middle| \begin{array}{l} K \in \mathfrak{o}(p-q+1) \\ L \in M(p-q+1,1;\mathbf{R}) \end{array} \right\}$$

$$\simeq \mathfrak{o}(p-q+1,1).$$

We express this subspace as $\mathfrak{g}_4$. Then, as in the case of $q =$ even, we can show that the bracket operations satisfy the following table:

[,]	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	$\mathfrak{g}_4$
$\mathfrak{g}_1$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	0
$\mathfrak{g}_2$		$\mathfrak{g}_1$	0	$\mathfrak{g}_2 + \mathfrak{g}_3$
$\mathfrak{g}_3$			$\mathfrak{g}_1$	$\mathfrak{g}_2$
$\mathfrak{g}_4$				$\mathfrak{g}_4$

Hence, $\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_4$ is a subalgebra of $\mathfrak{o}(p,q)$. In addition, the dimension of $\mathfrak{g}'$ is equal to the one given in Proposition 7 except the case $(p,q) = (3,3)$. Finally, the symplectic form ω on $\mathfrak{g}'$ can be defined in the same way as in the case of $q =$ even by adding the symplectic form ω' on $\mathfrak{g}_4$ constructed in Lemma 9 and the exact form $-d\alpha$ on $\mathfrak{g}'$, where $\alpha \in \mathfrak{g}'^*$ is defined by

$$\alpha\left(\left(\begin{array}{cccc} X^{(1)} & {}^tP & {}^tQ & X \\ -P & K & L & PE \\ Q & {}^tL & 0 & -QE \\ {}^tX & E^tP & E^tQ & X^{(2)} \end{array}\right)\right) = x_{12}^{(1)} + \cdots + x_{2r-1,2r}^{(1)},$$

as before. Verification of these facts can be done completely in the same way as before, and we leave it to the readers.

Finally, we construct a symplectic structure on a 12-dimensional subalgebra of $\mathfrak{o}(3,3)$ as follows. We put

$$\mathfrak{g}' = \left\{ \left(\begin{array}{cc} A & B \\ {}^tB & A+B-{}^tB \end{array} \right) \middle| A \in \mathfrak{o}(3),\ B \in M(3,3;\mathbf{R}) \right\}.$$

Then, $\mathfrak{g}'$ is a 12-dimensional subalgebra, and we define the element $\alpha \in \mathfrak{g}'^*$ by

$$\alpha(X) = x_{12} + x_{15} + x_{26},$$

where x_{ij} is the (i,j)-component of $X \in \mathfrak{g}'$. Then, it is directly checked that $-d\alpha$ gives a left invariant symplectic structure on $\mathfrak{g}'$, and we thus complete the proof of Proposition 7. ∎

Remark: The construction of the subalgebra of $\mathfrak{o}(p,q)$ stated in this proof is also based on the graded Lie algebra structure $\mathfrak{o}(p,q) = \mathfrak{l}_{-2} \oplus \cdots \oplus \mathfrak{l}_2$ with dim $\mathfrak{l}_{\pm 2} = 1$, as in the case of $\mathfrak{su}(p,q)$ (cf. [8], [14]). In the special case $\mathfrak{o}(3,3)$, it is isomorphic to $\mathfrak{sl}(4,\mathbf{R})$, and we already know that $\mathfrak{sl}(4,\mathbf{R})$ possesses a left invariant Poisson structure with rank $\pi = 12$. For general $\mathfrak{o}(p,p)$, we can construct the $1/2 \cdot p(3p-1)$-dimensional subalgebra of $\mathfrak{o}(p,p)$ by changing 3 into p in the above definition of $\mathfrak{g}' \subset \mathfrak{o}(3,3)$. The integer $1/2 \cdot p(3p-1)$ is even if and only if $p \equiv 0$ or 3 (mod 4), but it seems that this subalgebra does not possess a left invariant symplectic structure except for the case $p=3$. (In the case of $p=4$, we can show that the corresponding 22-dimensional subalgebra does not admit an "exact" left invariant symplectic structure.)

4. The case of $\mathrm{Sp}(p,q)$

In this section, we treat the remaining Grassmann type Lie group $G = \mathrm{Sp}(p,q)$ ($p \geq q \geq 1$). The construction is somewhat different from the previous cases $\mathrm{SU}(p,q)$ and $\mathrm{SO}(p,q)$. (See Remark after the proof of Proposition 10.)

PROPOSITION 10: *There exists a left invariant Poisson structure π on the Lie group $\mathrm{Sp}(p,q)$ $(p \geq q \geq 1)$ with*

$$\operatorname{rank}\pi = \begin{cases} 2p^2+2p & (p=q), \\ 2pq+p & (p = \textit{even} > q), \\ 2pq+p-1 & (p = \textit{odd} > q). \end{cases}$$

To prove this proposition, we first express the Lie algebra of $\mathrm{Sp}(p,q)$ in terms of quaternions $\mathbf{H}$ (cf. [12]). An element of $\mathbf{H}$ is expressed as $a_0 + a_1 i + a_2 j + a_3 k$ ($a_p \in \mathbf{R}$), and the product of i, j, k is given by the rule

$$\begin{gathered} i^2 = j^2 = k^2 = -1, \\ ij = k, \quad jk = i, \quad ki = j, \\ ji = -k, \quad kj = -i, \quad ik = -j. \end{gathered}$$

We imbed the field of real numbers $\mathbf{R}$ and the field of complex numbers $\mathbf{C}$ into $\mathbf{H}$ in a natural way. The conjugate of $w = a_0 + a_1 i + a_2 j + a_3 k \in \mathbf{H}$ is given by $\overline{w} = a_0 - a_1 i - a_2 j - a_3 k$. Then the following identity holds:

$$\overline{w_1 w_2} = \overline{w_2}\ \overline{w_1} \quad \text{for } w_1, w_2 \in \mathbf{H}.$$

Under these notations, we can express the Lie algebra of $\mathrm{Sp}(p,q)$ as

$$\mathfrak{sp}(p,q) = \{X \in M(p+q,p+q;\mathbf{H}) \mid {}^t\overline{X}I_{p,q} + I_{p,q}X = 0\}$$

$$= \left\{ \begin{array}{c} \begin{pmatrix} A & {}^t\overline{B} \\ B & C \end{pmatrix} \\ \begin{array}{cc} p & q \end{array} \end{array} \middle| \begin{array}{l} {}^t\overline{A} + A = 0, \quad {}^t\overline{C} + C = 0, \\ A \in M(p,p;\mathbf{H}),\ B \in M(q,p;\mathbf{H}),\ C \in M(q,q;\mathbf{H}) \end{array} \right\},$$

where

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

and I_p is the identity matrix of degree p. In general, an element $X \in M(n,n;\mathbf{H})$ can be uniquely expressed as $X = Y + jZ$ (Y, $Z \in M(n,n;\mathbf{C})$), and the map

$$Y + jZ \to \begin{pmatrix} Y & -\overline{Z} \\ Z & \overline{Y} \end{pmatrix}$$

gives a realization of the Lie algebra $M(n,n;\mathbf{H})$ in $M(2n,2n;\mathbf{C})$. (Note that $j\overline{z} = zj$ for $z \in \mathbf{C}$.) By this map, the Lie algebra $\mathfrak{sp}(p,q)$ is identified with the Lie algebra

$$\left\{ \begin{array}{cc} \begin{pmatrix} A_1 & {}^t\overline{B}_1 & -\overline{A}_2 & {}^t\overline{B}_2 \\ B_1 & C_1 & -\overline{B}_2 & -\overline{C}_2 \\ A_2 & -{}^tB_2 & \overline{A}_1 & {}^tB_1 \\ B_2 & C_2 & \overline{B}_1 & \overline{C}_1 \end{pmatrix} & \begin{array}{c} p \\ q \\ p \\ q \end{array} \\ \begin{array}{cccc} p & q & p & q \end{array} & \end{array} \middle| \begin{array}{ll} A_1 \in \mathfrak{u}(p), & C_1 \in \mathfrak{u}(q) \\ {}^tA_2 = A_2, & {}^tC_2 = C_2 \end{array} \right\}.$$

We remark that the definition of $\mathfrak{sp}(p,q)$ in [13; p. 446] is slightly different from the above. But it is easy to see that the two definitions coincide by the isomorphism given by $X \to P^{-1}XP$, where

$$P = \begin{pmatrix} I_p & & & 0 \\ & I_q & & \\ & & I_p & \\ 0 & & & -I_q \end{pmatrix}.$$

In the following, we assume that the Lie algebra $\mathfrak{sp}(p,q)$ is always realized in $M(p+q,p+q;\mathbf{H})$ in the above way.

Proof of Proposition 10: We first treat the case $p = q$. We define two subspaces $\mathfrak{g}_1$ and $\mathfrak{g}_2$ of $\mathfrak{g} = \mathfrak{sp}(p,p)$ by

$$\mathfrak{g}_1 = \left\{ \begin{array}{c} \begin{pmatrix} X & Xj \\ jX & jXj \end{pmatrix} \\ \begin{array}{cc} p & p \end{array} \end{array} \middle| {}^t\overline{X} + X = 0 \right\},$$

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \middle| D = \begin{pmatrix} d_1 i & & 0 \\ & \ddots & \\ 0 & & d_p i \end{pmatrix}, \ d_i \in \mathbf{R} \right\}.$$

We put $\mathfrak{g}' = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and express the element of $\mathfrak{g}'$ simply by the pair (X, D), where X and D are matrices appearing in the above definition of $\mathfrak{g}_i$. Then, we have

$$[(X_1, D_1), (X_2, D_2)] = ([X_1, D_2] - [X_2, D_1], 0) \in \mathfrak{g}_1,$$

and hence $\mathfrak{g}'$ is a $(2p^2+2p)$-dimensional subalgebra of $\mathfrak{sp}(p,p)$. We define elements of $\mathfrak{g}'$ by

$$\begin{aligned} &X_{ij} = (E_{ij} - E_{ji}, 0), \quad && Y_{ij} = (iE_{ij} + iE_{ji}, 0), \\ &Z_{ij} = (jE_{ij} + jE_{ji}, 0), && W_{ij} = (kE_{ij} + kE_{ji}, 0), \\ &D_i = (0, iE_{ii}), \end{aligned}$$

where E_{ij} is the matrix such that the (i,j)-component is 1 and other components are all zero. Then, the bracket operations of these elements are given by

$$\begin{aligned} [X_{ij}, D_k] &= \delta_{jk} Y_{ki} - \delta_{ik} Y_{kj}, \\ [Y_{ij}, D_k] &= \delta_{jk} X_{ki} + \delta_{ik} X_{kj}, \\ [Z_{ij}, D_k] &= -\delta_{jk} W_{ki} - \delta_{ik} W_{kj}, \\ [W_{ij}, D_k] &= \delta_{jk} Z_{ki} + \delta_{ik} Z_{kj}, \end{aligned}$$

and the remaining brackets are all zero. The elements

$$\{X_{ij}\}_{1 \le i < j \le p} \cup \{Y_{ij}, Z_{ij}, W_{ij}\}_{1 \le i \le j \le p} \cup \{D_i\}_{1 \le i \le p}$$

form a basis of $\mathfrak{g}'$, and we denote its dual basis by α_{ij}, β_{ij}, γ_{ij}, ε_{ij}, μ_i, respectively. Then, from the above bracket table, we have

$$\begin{aligned} d\alpha_{ij} &= \beta_{ij} \wedge (\mu_j - \mu_i) \quad && (i < j), \\ d\beta_{ij} &= \alpha_{ij} \wedge (\mu_i - \mu_j) && (i < j), \\ d\beta_{ii} &= 0, \\ d\gamma_{ij} &= -\varepsilon_{ij} \wedge (\mu_i + \mu_j) && (i \le j), \\ d\varepsilon_{ij} &= \gamma_{ij} \wedge (\mu_i + \mu_j) && (i \le j), \\ d\mu_i &= 0. \end{aligned}$$

Hence, the 2-form $\omega = \sum_{i<j} \alpha_{ij} \wedge \beta_{ij} + \sum_{i \le j} \gamma_{ij} \wedge \varepsilon_{ij} + \sum_i \beta_{ii} \wedge \mu_i$ is non-degenerate and closed, which gives a desired left invariant symplectic structure on $\mathfrak{g}'$.

Next, we treat the case $p > q$. In this case, we express an element of $\mathfrak{sp}(p,q) \subset M(p+q, p+q; \mathbf{H})$ as

$$\begin{array}{cl} \begin{pmatrix} A & -{}^t\overline{B} & {}^t\overline{C} \\ B & D & {}^t\overline{E} \\ C & E & F \end{pmatrix} & \begin{array}{l} q \\ p-q \\ q \end{array} \\ \begin{array}{ccc} q & p-q & q \end{array} & \end{array}, \qquad \begin{array}{l} {}^t\overline{A}+A=0, \\ {}^t\overline{D}+D=0, \\ {}^t\overline{F}+F=0. \end{array}$$

Using this notation, we construct a subalgebra $\mathfrak{g}'$ of $\mathfrak{sp}(p,q)$ in the following way. We define the elements of $\mathfrak{sp}(p,q)$ by

$$X_{ij} = \begin{pmatrix} i(E_{ij}+E_{ji}) & 0 & k(E_{ij}+E_{ji}) \\ 0 & 0 & 0 \\ -k(E_{ij}+E_{ji}) & 0 & i(E_{ij}+E_{ji}) \end{pmatrix},$$

$$X'_{ij} = \begin{pmatrix} E_{ij}-E_{ji} & 0 & j(E_{ij}-E_{ji}) \\ 0 & 0 & 0 \\ j(E_{ij}-E_{ji}) & 0 & -(E_{ij}-E_{ji}) \end{pmatrix},$$

$$Y_{ij} = \begin{pmatrix} E_{ij}-E_{ji} & 0 & j(E_{ij}+E_{ji}) \\ 0 & 0 & 0 \\ -j(E_{ij}+E_{ji}) & 0 & E_{ij}-E_{ji} \end{pmatrix},$$

$$Y'_{ij} = \begin{pmatrix} i(E_{ij}+E_{ji}) & 0 & k(E_{ij}-E_{ji}) \\ 0 & 0 & 0 \\ k(E_{ij}-E_{ji}) & 0 & -i(E_{ij}+E_{ji}) \end{pmatrix},$$

$$Z_{ij} = \begin{pmatrix} 0 & -E_{ji} & 0 \\ E_{ij} & 0 & jE_{ij} \\ 0 & -jE_{ji} & 0 \end{pmatrix}, \quad Z'_{ij} = \begin{pmatrix} 0 & iE_{ji} & 0 \\ iE_{ij} & 0 & kE_{ij} \\ 0 & -kE_{ji} & 0 \end{pmatrix},$$

$$W_i = \begin{pmatrix} iE_{ii} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -iE_{ii} \end{pmatrix}, \qquad D_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & iE_{ii} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that the spaces $\{X_{ij}\ (1 \le i \le j \le q),\ X'_{ij}\ (1 \le i < j \le q)\}$, $\{Z_{ij}, Z'_{ij}(1 \le i \le p-q,\ 1 \le j \le q)\}$ just coincide with the spaces

$$\left\{ \begin{pmatrix} X & 0 & Xj \\ 0 & 0 & 0 \\ jX & 0 & jXj \end{pmatrix} \middle| X \in \mathfrak{u}(q) \right\},$$

$$\left\{ \begin{pmatrix} 0 & -{}^t\overline{A} & 0 \\ A & 0 & Aj \\ 0 & -j{}^t\overline{A} & 0 \end{pmatrix} \middle| A \in M(p-q,q;\mathbf{C}) \right\},$$

respectively. The bracket operations of these matrices are given by

$$
\begin{aligned}
&[X_{ij}, Y_{kl}] = 2\delta_{jk}X_{il} + 2\delta_{ik}X_{jl},\\
&[X_{ij}, Y'_{kl}] = -2\delta_{jk}X'_{il} - 2\delta_{ik}X'_{jl},\\
&[X'_{ij}, Y_{kl}] = 2\delta_{jk}X'_{il} - 2\delta_{ik}X'_{jl},\\
&[X'_{ij}, Y'_{kl}] = 2\delta_{jk}X_{il} - 2\delta_{ik}X_{jl},\\
&[X_{ij}, W_k] = -\delta_{jk}X'_{ik} - \delta_{ik}X'_{jk},\\
&[X'_{ij}, W_k] = \delta_{jk}X_{ik} - \delta_{ik}X_{jk},\\
&[Y_{ij}, Y_{kl}] = 2\delta_{jk}Y_{il} - 2\delta_{il}Y_{kj},\\
&[Y_{ij}, Y'_{kl}] = 2\delta_{jk}Y'_{il} - 2\delta_{il}Y'_{kj},\\
&[Y'_{ij}, Y'_{kl}] = -2\delta_{jk}Y_{il} + 2\delta_{il}Y_{kj},\\
&[Y_{ij}, Z_{kl}] = -[Y'_{ij}, Z'_{kl}] = -2\delta_{il}Z_{kj},\\
&[Y_{ij}, Z'_{kl}] = [Y'_{ij}, Z_{kl}] = -2\delta_{il}Z'_{kj},\\
&[Y_{ij}, W_k] = \delta_{jk}Y'_{ik} - \delta_{ik}Y'_{kj},\\
&[Y'_{ij}, W_k] = -\delta_{jk}Y_{ik} + \delta_{ik}Y_{kj},\\
&[Z_{ij}, Z_{kl}] = [Z'_{ij}, Z'_{kl}] = -\delta_{ik}X'_{jl},\\
&[Z_{ij}, Z'_{kl}] = -\delta_{ik}X_{jl},\\
&[Z_{ij}, W_k] = \delta_{jk}Z'_{ik}, \quad [Z'_{ij}, W_k] = -\delta_{jk}Z_{ik},\\
&[Z_{ij}, D_k] = -\delta_{ik}Z'_{kj}, \quad [Z'_{ij}, D_k] = \delta_{ik}Z_{kj},
\end{aligned}
$$

and the remaining brackets are all zero. Using these relations, we know that the space $\mathfrak{g}'$ spanned by the matrices

$$
\begin{array}{ll}
X_{ij}\ (1 \le i \le j \le q), & X'_{ij}\ (1 \le i < j \le q),\\
Y_{ij}\ (1 \le i \le j \le q), & Y'_{ij}\ (1 \le i < j \le q),\\
Z_{ij}\ (1 \le i \le p-q,\ 1 \le j \le q), & Z'_{ij}\ (1 \le i \le p-q,\ 1 \le j \le q),\\
W_i\ (1 \le i \le q), & D_i\ (1 \le i \le p-q)
\end{array}
$$

forms a $(2pq+p)$-dimensional subalgebra $\mathfrak{g}'$ of $\mathfrak{sp}(p,q)$. (We remark that $X_{ij} = X_{ji}$ and $X'_{ij} = -X'_{ji}$, but there is no linear relation between Y_{ij}, Y_{ji}, Y'_{ij}, Y'_{ji}, and $\mathfrak{g}'$ does not contain elements $Y_{ij}\,(i>j)$, $Y'_{ij}\,(i \ge j)$.) We denote by α_{ij}, α'_{ij}, β_{ij}, β'_{ij}, γ_{ij}, γ'_{ij}, ε_i, μ_i the dual basis of $\mathfrak{g}'^*$. Then, from the above bracket relations, we have

$$
d\alpha_{ii} = -2\sum_{j=1}^{i-1}\alpha_{ij}\wedge\beta_{ji} - 4\alpha_{ii}\wedge\beta_{ii} - 2\sum_{j=1}^{i-1}\alpha'_{ij}\wedge\beta'_{ji} + \sum_{j=1}^{p-q}\gamma_{ji}\wedge\gamma'_{ji}
$$

and $d\varepsilon_1 = \cdots = d\varepsilon_q = d\mu_1 = \cdots = d\mu_{p-q} = 0$. Hence, in the case $p =$ even, the 2-form

$$
\omega = -2\sum_{j<i}\alpha_{ij}\wedge\beta_{ji} - 4\sum_{i}\alpha_{ii}\wedge\beta_{ii} - 2\sum_{j<i}\alpha'_{ij}\wedge\beta'_{ji} + \sum_{j=1}^{p-q}\sum_{i=1}^{q}\gamma_{ji}\wedge\gamma'_{ji} + \sum_{i=1}^{p/2}\nu_{2i-1}\wedge\nu_{2i}
$$

gives the desired symplectic form on $\mathfrak{g}'$, where $\nu_1 = \varepsilon_1, \dots, \nu_q = \varepsilon_q$, $\nu_{q+1} = \mu_1, \dots, \nu_p = \mu_{p-q}$. In the case $p =$ odd, we can construct the subalgebra and the symplectic form on it in exactly the same way by deleting the last matrix D_{p-q} from the above construction. ∎

Remark: As in the case of $\mathfrak{su}(p,q)$ in §2, we can construct a subalgebra of $\mathfrak{sp}(p,q)$ with dimension $4pq+2[(p-q)/2]$ in exactly the same way by changing the field $\mathbf{C}$ into $\mathbf{H}$. But it seems that this subalgebra does not admit a left invariant symplectic structure, and we must adopt a different construction than above. (We can verify this conjecture for the Lie algebras $\mathfrak{sp}(1,1)$, $\mathfrak{sp}(2,1)$, $\mathfrak{sp}(3,1)$ and $\mathfrak{sp}(2,2)$, though the Lie algebra $\mathfrak{sp}(1,1) \simeq \mathfrak{o}(4,1)$ admits another type of 4-dimensional subalgebra possessing a left invariant symplectic structure; cf. Proposition 7.) This difference comes from the fact that general $\mathfrak{sp}(p,q)$ does not admit a structure of a graded Lie algebra of the second kind satisfying the condition $\dim \mathfrak{l}_{\pm 2} = 1$ in contrast with $\mathfrak{su}(p,q)$ and $\mathfrak{o}(p,q)$ (cf. [8], [14]).

5. The cases of $\mathrm{SU}^*(2n)$, $\mathrm{Sp}(n,\mathbf{R})$ and $\mathrm{SO}^*(2n)$

In this section, we treat the remaining non-compact classical simple Lie groups of non-Grassmann type.

PROPOSITION 11: *There exists a left invariant Poisson structure π on the Lie group $\mathrm{SU}^*(2n)$ $(n \geq 2)$ with $\operatorname{rank}\pi = 4n(n-1)$.*

Proof: The Lie algebra of $\mathrm{SU}^*(2n)$ is given by

$$\mathfrak{su}^*(2n) = \left\{ \left(\begin{array}{cc} A & B \\ -\overline{B} & \overline{A} \end{array} \right) \middle| A, B \in M(n,n;\mathbf{C}),\ \ \operatorname{Tr} A \in i\mathbf{R} \right\}.$$

We express the above matrix simply as $(A,\ B)$. Under this notation, we consider the subspace $\mathfrak{g}'$ of $\mathfrak{su}^*(2n)$ consisting of matrices $(A,\ B)$ such that $a_{ni} = b_{nj} = 0$ for $1 \leq i \leq n-1$, $1 \leq j \leq n$, and $a_{nn} = -\mathrm{Re}\,(a_{11} + \cdots + a_{n-1,n-1})$, where a_{ij} and b_{ij} are the (i,j)-components of A and B, respectively. Then, we can easily check that $\mathfrak{g}'$ is a $4n(n-1)$-dimensional subalgebra of $\mathfrak{su}^*(2n)$. Next, we define the element $\alpha \in \mathfrak{g}'^*$ by

$$\alpha((A,\ B)) = a_{12} + a_{23} + \cdots + a_{n-1,n},$$

and put $\omega = -d\alpha$. Then, as in the case of $\mathrm{SL}(n,\mathbf{R})$ [2; p. 180] (or §7 (2)), we can show that ω is non-degenerate, and hence it gives a symplectic structure on $\mathfrak{g}'$. We leave detailed calculations to the readers. ∎

Remark: In the case of $n=2$, the Lie algebra $\mathfrak{su}^*(4)$ is isomorphic to $\mathfrak{o}(5,1)$, and the above subalgebra $\mathfrak{g}'$ corresponds to the subalgebra of $\mathfrak{o}(5,1)$ constructed in Proposition 7.

PROPOSITION 12: *There exists a left invariant Poisson structure π on the Lie group $\mathrm{Sp}(n,\mathbf{R})$ with $\operatorname{rank}\pi = n(n+1)$.*

Proof: We remember that the Lie algebra of $\mathrm{Sp}(n,\mathbf{R})$ is given by

$$\mathfrak{sp}(n,\mathbf{R}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \middle| A,\, B,\, C \in M(n,n;\mathbf{R}),\ {}^tB = B,\ {}^tC = C \right\}.$$

Using this notation, we put

$$\mathfrak{g}' = \left\{ \begin{pmatrix} A & B \\ 0 & -{}^tA \end{pmatrix} \middle| A \text{ is upper triangular, } {}^tB = B \right\}.$$

Then, $\mathfrak{g}'$ is an $n(n+1)$-dimensional subalgebra of $\mathfrak{sp}(n,\mathbf{R})$. Next, we define $\alpha \in \mathfrak{g}'^*$ by

$$\alpha\left(\begin{pmatrix} A & B \\ 0 & -{}^tA \end{pmatrix}\right) = \operatorname{Tr} B,$$

and put $\omega = -d\alpha$. Then, it is easy to check that ω gives the desired symplectic form on $\mathfrak{g}'$. ∎

PROPOSITION 13: *There exists a left invariant Poisson structure π on the Lie group $\mathrm{SO}^*(2n)$ with*

$$\operatorname{rank}\pi = \begin{cases} n(n-1) & (n \geq 2,\, n \neq 4), \\ 14 & (n=4). \end{cases}$$

The Lie algebra $\mathfrak{o}^*(8)$ is isomorphic to $\mathfrak{o}(6,2)$, and hence from Proposition 7, we know that $\mathfrak{o}^*(8)$ possesses a Poisson structure π with $\operatorname{rank}\pi = 14$.

To prove the proposition for general case, we first define four real vector spaces by

$$M_1 = \left\{ p \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \middle| p \in \mathbf{R} \right\},$$

$$M_2 = \left\{ p \begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix} \middle| p \in \mathbf{R} \right\},$$

$$M_3 = \left\{ \begin{pmatrix} a & b \\ ai & bi \end{pmatrix} \middle| a,\, b \in \mathbf{C} \right\},$$

$$M_4 = \left\{ \begin{pmatrix} a \\ ai \end{pmatrix} \middle| a \in \mathbf{C} \right\}.$$

Clearly, M_2 is a subspace of M_3. Then, these spaces possess the following properties. The verification of these facts are easy.

LEMMA 14: *Assume $X_i,\, Y_i \in M_i$ $(i = 1 \sim 4)$. Then, we have*

$$X_2\overline{Y}_2 = X_2\overline{Y}_3 = X_2\overline{Y}_4 = {}^tX_3Y_3 = {}^tX_3Y_4 = {}^tX_4Y_4 = 0,$$

$$X_1Y_2,\ X_2Y_1,\ X_3{}^t\overline{Y}_3 + Y_3{}^t\overline{X}_3,\ X_4{}^t\overline{Y}_4 + Y_4{}^t\overline{X}_4 \in M_2,$$

$$X_1Y_3,\ X_3Y_1,\ X_3Y_2,\ X_3\overline{Y}_2,\ X_3Y_3,\ X_3\overline{Y}_3,\ X_3{}^t\overline{Y}_3,\ X_4{}^t\overline{Y}_4 \in M_3,$$

$$X_3{}^tY_3 = Y_3{}^tX_3 \in M_3,\ X_4{}^tY_4 = Y_4{}^tX_4 \in M_3,$$

$$X_1Y_4,\ X_3Y_4,\ X_3\overline{Y}_4 \in M_4.$$

Proof of Proposition 13: We assume $n \geq 2$ and $n \neq 4$. The Lie algebra of $\mathrm{SO}^*(2n)$ is given by

$$\mathfrak{o}^*(2n) = \left\{ \begin{pmatrix} A & B \\ -\overline{B} & \overline{A} \end{pmatrix} \middle| A \in \mathfrak{o}(n,\mathbf{C}),\ {}^tB = \overline{B} \in M(n,n;\mathbf{C}) \right\},$$

and we express the above element of $\mathfrak{o}^*(2n)$ simply as $(A,\, B)$. We first consider the case $n = 2r$. We express the matrices A and B in terms of (2,2)-blocks as follows:

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1r} \\ & \cdots & \\ A_{r1} & \cdots & A_{rr} \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & \cdots & B_{1r} \\ & \cdots & \\ B_{r1} & \cdots & B_{rr} \end{pmatrix},$$

where $A_{ij},\, B_{ij} \in M(2,2;\mathbf{C})$. Clearly, we have

$$A_{ji} = -{}^tA_{ij}, \quad B_{ji} = {}^t\overline{B}_{ij}, \quad 1 \leq i < j \leq r,$$

and $A_{ii} \in \mathfrak{o}(2, \mathbf{C})$, ${}^tB_{ii} = \overline{B}_{ii}$, $1 \le i \le r$. Under these notations, we consider the subspace $\mathfrak{g}'$ of $\mathfrak{o}^*(2n)$ consisting of the pair $(A,\, B)$ which satisfies

$$\begin{array}{ll} A_{ii} \in M_1,\ B_{ii} \in M_2, & 1 \le i \le r, \\ A_{ij},\, B_{ij} \in M_3, & 1 \le i < j \le r. \end{array}$$

Then, by using the properties in Lemma 14, and the formula

$$[(A,\, B),\, (C,\, D)] = ([A,\, C] - B\overline{D} + D\overline{B}, AD - D\overline{A} + B\overline{C} - CB),$$

we can directly check that $\mathfrak{g}'$ is the $n(n-1)$-dimensional subalgebra of $\mathfrak{o}^*(2n)$. For example, for $i < j$, we have

$$\begin{aligned} [A, C]_{ij} \quad &= \quad \sum_{k<i}(-{}^tA_{ki}C_{kj} + {}^tC_{ki}A_{kj}) + A_{ii}C_{ij} - C_{ii}A_{ij} \\ &\quad + \sum_{i<k<j}(A_{ik}C_{kj} - C_{ik}A_{kj}) + A_{ij}C_{jj} - C_{ij}A_{jj} \\ &\quad + \sum_{j<k}(-A_{ik}{}^tC_{jk} + C_{ik}{}^tA_{jk}) \in M_3 \end{aligned}$$

on account of the property ${}^tX_3Y_3 = 0$ and X_1Y_3, X_3Y_3, X_3Y_1, $X_3{}^tY_3 \in M_3$. Next, we define the element α of $\mathfrak{g}'^*$ by

$$\alpha((A,\, B)) = \operatorname{Tr} B,$$

and put $\omega = -d\alpha$. Then, we can easily show that ω gives the desired symplectic structure on $\mathfrak{g}'$.

Next, we consider the case $n = 2r + 1$. In this case, we express A and B as

$$A = \begin{pmatrix} A_{11} & \cdots & A_{1r} & \alpha_1 \\ & \cdots & & \\ A_{r1} & \cdots & A_{rr} & \alpha_r \\ -{}^t\alpha_1 & \cdots & -{}^t\alpha_r & 0 \end{pmatrix}, \quad B = \begin{pmatrix} B_{11} & \cdots & B_{1r} & \beta_1 \\ & \cdots & & \\ B_{r1} & \cdots & B_{rr} & \beta_r \\ {}^t\overline{\beta}_1 & \cdots & {}^t\overline{\beta}_r & b \end{pmatrix},$$

where α_i, $\beta_i \in M(2, 1; \mathbf{C})$, and $b \in \mathbf{R}$. Under these notations, we consider the subspace $\mathfrak{g}'$ of $\mathfrak{o}^*(2n)$ consisting of the pair $(A,\, B)$ which satisfies

$$\begin{array}{ll} A_{ii} \in M_1,\ B_{ii} \in M_2, & 1 \le i \le r, \\ A_{ij},\, B_{ij} \in M_3, & 1 \le i < j \le r, \\ \alpha_i,\, \beta_i \in M_4, & 1 \le i \le r, \\ b = 0. & \end{array}$$

Then, by using Lemma 14 again, we can show that $\mathfrak{g}'$ is the $n(n-1)$-dimensional subalgebra of $\mathfrak{o}^*(2n)$. The symplectic structure on $\mathfrak{g}'$ can be constructed in exactly the same way as in the case of $n = 2r$, and we leave detailed calculations to the readers. ∎

6. The case of complex simple Lie groups

Finally, we treat the case of complex simple Lie groups, considered as real Lie groups. We first prepare the following general result.

PROPOSITION 15: *Let $\mathfrak{h}$ be a real Lie algebra with a left invariant symplectic structure. Then, the complexification $\mathfrak{h}^{\mathbf{C}}$ of $\mathfrak{h}$ considered as a real Lie algebra also possesses a left invariant symplectic structure.*

Proof: By using the symplectic structure ω on $\mathfrak{h}$, we define the $\mathbf{R}$-valued skew-symmetric map ω' on $\mathfrak{h}^{\mathbf{C}}$ by

$$\omega'(X_1 + iY_1,\, X_2 + iY_2) = \omega(X_1,\, X_2) - \omega(Y_1,\, Y_2),$$

for $X_i,\, Y_i \in \mathfrak{h}$. Then, it is easy to see that ω' is closed and non-degenerate on $\mathfrak{h}^{\mathbf{C}}$, and this gives the desired symplectic form. ∎

By this proposition, we know that if a real Lie algebra $\mathfrak{g}$ possesses a left invariant Poisson structure with $\operatorname{rank}\pi = 2k$, then its complexification $\mathfrak{g}^{\mathbf{C}}$ considered as a real Lie algebra admits a left invariant Poisson structure with $\operatorname{rank}\pi = 4k$.

Applying this proposition to each real form of complex simple Lie algebras, we obtain the following proposition.

PROPOSITION 16: *Classical complex simple Lie groups considered as real Lie groups possess a left invariant Poisson structure with the following rank:*

$$\begin{aligned}
\mathrm{SL}(n,\mathbf{C})^{\mathbf{R}}&\colon\ \operatorname{rank}\pi = 2n(n-1),\\
\mathrm{O}(2n+1,\mathbf{C})^{\mathbf{R}}&\colon\ \operatorname{rank}\pi = 2n(n+1),\\
\mathrm{Sp}(n,\mathbf{C})^{\mathbf{R}}&\colon\ \operatorname{rank}\pi = 2n(n+1),\\
\mathrm{O}(2n,\mathbf{C})^{\mathbf{R}}&\colon\ \operatorname{rank}\pi = \begin{cases} 2n^2 & (n = \mathit{even}),\\ 2(n^2-1) & (n = \mathit{odd}, n \neq 3),\\ 24 & (n = 3). \end{cases}
\end{aligned}$$

To prove this proposition, we have only to find the highest rank solutions π constructed in the previous sections for all real forms of each complex simple Lie algebra. We only exhibit the list of real forms giving the highest rank solution:

$$
\begin{aligned}
\mathfrak{sl}(n,\mathbf{C})&\colon \mathfrak{sl}(n,\mathbf{R}),\\
\mathfrak{o}(2n+1,\mathbf{C})&\colon \mathfrak{o}(n+1,n),\\
\mathfrak{sp}(n,\mathbf{C})&\colon \mathfrak{sp}(n,\mathbf{R}),\\
\mathfrak{o}(2n,\mathbf{C})&\colon \begin{cases} \mathfrak{o}(n+1,n-1), \mathfrak{o}(n,n) & (n=\text{even}),\\ \mathfrak{o}(n+2,n-2),\ \mathfrak{o}(n+1,n-1),\ \mathfrak{o}(n,n) & (n=\text{odd}, n\neq 3),\\ \mathfrak{o}(3,3) & (n=3).\end{cases}
\end{aligned}
$$

Detailed examination of this fact is easy and left to the readers.

Thus, combining the propositions in §§2–6, we complete the proof of Theorem 1.

7. Final remarks

In this final section, we state some results and comments related to left invariant Poisson structures.

(1) Poisson–Lie groups.

We say that a Lie group G with a Poisson structure $\{\ ,\ \}$ is a Poisson–Lie group if the multiplication map $G\times G\to G$ is a Poisson map, where $G\times G$ is endowed with the product Poisson structure (cf. [10], [24]). In the following, we regard the element $\pi\in\wedge^2\mathfrak{g}$ as the 2-vector at the identity element of G, and we denote by $\overline{\pi}$ (resp. $\widetilde{\pi}$) the left invariant (resp. right invariant) 2-vector field extended to the whole space G by the group action. Drinfel'd [10] proved that $\overline{\pi}-\widetilde{\pi}$ gives a Poisson–Lie group structure on G if and only if $[\pi,\pi]_S$ is invariant under the adjoint action of G. (See also [24; p. 173].) In addition, in [18], it is shown that if G is connected and semi-simple, or if G is compact, then all Poisson–Lie group structures on G are expressed in this form $\overline{\pi}-\widetilde{\pi}$. In particular, from these results, we know that the solutions of the CYB-equation $[\pi,\pi]_S=0$ constructed in this paper give a new class of Poisson–Lie group structures on G because $[\pi,\pi]_S$ is clearly $\operatorname{Ad}G$-invariant in this case. The 2-vector $\pi\in\wedge^2\mathfrak{g}$ with $\operatorname{Ad}G$-invariant $[\pi,\pi]_S$ ($\neq 0$) is studied in detail and classified in [3], [7] (and [25] for the case $\mathfrak{g}=\mathfrak{o}(3,1)$), though there was no detailed study of the solution of the CYB-equation $[\pi,\pi]_S=0$ itself except some special cases, as we stated before.

(2) Solutions of the CYB-equation for $G=\mathrm{SL}(n,\mathbf{R})$.

At the end of the paper [2], Belavin and Drinfel'd constructed several solutions of the CYB-equation for the groups $\mathrm{GL}(n,\mathbf{C})$ and $\mathrm{SL}(n,\mathbf{C})$. Their construction is also valid in the real case, and the highest rank solutions among them are of rank $n(n-1)$ for both Lie groups. For example, in the case of $\mathfrak{g}=\mathfrak{gl}(n,\mathbf{R})$, we define the $n(n-1)$-dimensional subalgebra $\mathfrak{g}'$ by

$$\mathfrak{g}' = \left\{ \left(\begin{array}{cc} A & v \\ 0 & 0 \end{array} \right) \middle| A \in M(n-1, n-1; \mathbf{R}), \;\; v \in M(n-1, 1; \mathbf{R}) \right\},$$

and the element $\alpha \in \mathfrak{g}'^*$ by

$$\alpha(X) = x_{12} + x_{23} + \cdots + x_{n-1,n}$$

for $X = (x_{ij}) \in \mathfrak{g}'$. (Note that $\mathfrak{g}'$ is naturally isomorphic to the affine Lie algebra $\mathfrak{a}(n-1, \mathbf{R})$.) Then, it is easy to check that the exact 2-form $-d\alpha$ gives the symplectic form on $\mathfrak{g}'$, and the corresponding 2-vector π is expressed as

$$\pi = \sum_{1 \leq i \leq j \leq k \leq n-1} E_{ji} \wedge E_{k-j+i,k+1}.$$

The $n(n-1)$-dimensional subalgebra of $\mathfrak{sl}(n, \mathbf{R})$, the symplectic form on it, and the 2-vector π for $\mathfrak{sl}(n, \mathbf{R})$ can be obtained from this example by projecting them to $\mathfrak{sl}(n, \mathbf{R})$. In particular, we have

$$\pi = \sum_{1 \leq i \leq j \leq k \leq n-1} (E_{ji} - 1/n \cdot \delta_{ji} I_n) \wedge E_{k-j+i,k+1}$$

for $\mathfrak{sl}(n, \mathbf{R})$, where I_n is the identity matrix of degree n. (Similar results can be found in several references such as [5], [19], [20], etc. These solutions satisfy the equality $\operatorname{rank} \pi = \dim G - \operatorname{rank} G$, and are relatively high rank solutions compared with other non-compact Lie groups in Theorem 1. See (5) below.)

For both Lie groups $\mathrm{GL}(n, \mathbf{R})$ and $\mathrm{SL}(n, \mathbf{R})$, by applying the result of Dynkin [11], we can prove that the examples in [2] give the maximum of $\operatorname{rank} \pi$ among the solutions of the CYB-equation. We can also show that for the group $\mathrm{SL}(n, \mathbf{R})$ ($n \geq 2$) the solutions of the CYB-equation with rank $\pi = n(n-1)$ are uniquely determined under the action of the automorphism group of $\mathfrak{sl}(n, \mathbf{R})$. For details, see [1].

The classification of the solutions for $\mathrm{GL}(2, \mathbf{R})$ is given in [17; p. 36], and we can further show that the solutions are essentially deformable in contrast with the group $\mathrm{SL}(n, \mathbf{R})$. See also (4) below.

(3) Maximum value of rank π for low dimensional Lie groups.

At present, as for other non-compact simple real Lie groups, the maximum of rank π is determined only in the following cases, as stated in Introduction:

G	max rank π	maximum subalgebra
$\mathrm{SU}(2,1)$	4	$\dim = 5$
$\mathrm{SO}(3,1) \sim \mathrm{SL}(2,\mathbf{C})^{\mathbf{R}}$	4	$\dim = 4$
$\mathrm{SO}(4,1) \sim \mathrm{Sp}(1,1)$	4	$\dim = 7$
$\mathrm{SO}(3,2) \sim \mathrm{Sp}(2,\mathbf{R})$	6	$\dim = 7$
$\mathrm{SO}(5,1) \sim \mathrm{SU}^*(4)$	8	$\dim = 11$

For example, it is known that the maximum dimension of proper subalgebras of $\mathfrak{su}(2,1)$ is 5 (cf. [22; p. 1390]). Hence, we have max rank $\pi = 4$ for $\mathrm{SU}(2,1)$, as a trivial consequence of Theorem 1 and Proposition 2. In the same way, we can show the above result for $\mathrm{SO}(3,2)$ because the maximum dimension of proper subalgebras of $\mathfrak{o}(p,q)$ is equal to $1/2 \cdot \{(p+q)^2 - 3(p+q) + 4\}$ for $p+q \geq 3$ and $p+q \neq 4$, 6. (This fact follows immediately from the result of Dynkin [11]. From his result, we can show that the maximum dimension of complex proper subalgebras of the complex Lie algebra $\mathfrak{o}(n,\mathbf{C})$ is $1/2 \cdot (n^2 - 3n + 4)$ for $n \geq 3$ and $n \neq 4$, 6. Existence of the above dimensional real subalgebra of $\mathfrak{o}(p,q)$ is shown by taking the non-negative part of the graded Lie algebra structure of the first kind $\mathfrak{o}(p,q) = \mathfrak{l}_{-1} \oplus \mathfrak{l}_0 \oplus \mathfrak{l}_1$ (cf. [15]).) For the Lie algebras $\mathfrak{o}(3,1)$ and $\mathfrak{o}(5,1)$, we can show that the maximum dimensions of proper subalgebras are 4 and 11, respectively, by classifying high dimensional subalgebras, and the above result for $\mathrm{SO}(3,1)$ follows immediately from this fact. But for two Lie algebras $\mathfrak{o}(4,1)$ and $\mathfrak{o}(5,1)$, there are several 6- and 10-dimensional subalgebras, and hence we must classify such dimensional subalgebras and must show that these subalgebras do not possess left invariant symplectic structures.

For other remaining high dimensional Lie groups, we cannot carry out such a procedure any more. But, it seems to the author that the values in Theorem 1 give the maximum of rank π for most Lie groups.

(4) Algebraic sets defined by the CYB-equation.

In terms of a basis $\{X_i\}$ of $\mathfrak{g}$, we express the bracket operation of $\mathfrak{g}$ by $[X_i, X_j] = \sum c_{ij}^k X_k$, and express π as $\sum a_{ij} X_i \wedge X_j$. Then, up to a non-zero constant, the Schouten bracket $[\pi,\pi]_S$ is equal to

$$\sum (a_{iq}a_{jr}c_{ij}^p + a_{ir}a_{jp}c_{ij}^q + a_{ip}a_{jq}c_{ij}^r) X_p \wedge X_q \wedge X_r \in \wedge^3 \mathfrak{g},$$

and hence the condition $[\pi,\pi]_S = 0$ is equivalent to $[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0$ stated in Introduction. (Compare the equation in [2; p. 162].) Thus, in order to obtain left invariant Poisson structures on G, we have only to solve this system of quadratic equations, and for small dimensional Lie groups, we can obtain all solutions of the CYB-equation in this way. For example, in the case $G = \mathrm{GL}(2,\mathbf{R})$, we can show that the set of solutions is decomposed into a union of two irreducible 3-dimensional varieties (cf. [17; p. 36]), one of which is the 3-dimensional plane in $\wedge^2\mathfrak{gl}(2,\mathbf{R}) \simeq \mathbf{R}^6$ consisting of 2-vectors of the form $X \wedge (E_{11}+E_{22})$ $(X \in \mathfrak{gl}(2,\mathbf{R}))$, and the other of which is the non-linear variety consisting of the elements

$$\begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \wedge \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

$$\pm\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \wedge \begin{pmatrix} d & -b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \wedge \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with $a^2+bc=0$. In the case of $\mathrm{SL}(2,\mathbf{R})$, the solutions constitute an irreducible quadratic cone in $\wedge^2\mathfrak{sl}(2,\mathbf{R}) \simeq \mathbf{R}^3$ (cf. [21]), which consists of three adjoint orbits including the trivial one. But unfortunately, for higher dimensional Lie groups, we cannot solve the system of quadratic equations in an explicit form, or even determine whether the algebraic set defined by $[\pi,\pi]_S = 0$ is irreducible or not.

(5) An upper bound on rank π for exact symplectic structures.
In Proposition 2, we now restrict ourselves to the "exact" symplectic structure on $\mathfrak{g}'$. Then we have the following restricted "upper bound" on max rank π.

PROPOSITION 17: *Assume that the Lie algebra $\mathfrak{g}$ of G is semi-simple, and a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ admits a left invariant "exact" symplectic structure. Then, the inequality* $\dim \mathfrak{g}' \leq \dim G - \operatorname{rank} G$ *holds.*

Remark: As stated in (2), the equality in this proposition holds for $\mathrm{SL}(n,\mathbf{R})$. (The symplectic structure constructed in [2] was exact.) But the value $\dim G - \operatorname{rank} G$ does not give the actual upper bound of rank π for general non-compact Lie groups. Remember the examples in (3) above.

To prove Proposition 17, we use the following well-known result.

LEMMA 18: *Let $\mathfrak{h}$ be the Lie algebra of a Lie group H. Then, for $\alpha \in \mathfrak{h}^*$, the exact 2-form $d\alpha$ gives a left invariant symplectic structure on H if and only if $\mathcal{O}_H(\alpha)$ is open in $\mathfrak{h}^*$, where $\mathcal{O}_H(\alpha)$ is the coadjoint H-orbit of α.*

For the proof of this lemma, see for example [19].

Proof of Proposition 17: Let $\omega = d\alpha$ $(\alpha \in \mathfrak{g}'^*)$ be a symplectic structure on G', where G' is the Lie subgroup of G with Lie algebra $\mathfrak{g}'$. Then, since the dual map $i^*\colon \mathfrak{g}^* \to \mathfrak{g}'^*$ of the inclusion $i\colon \mathfrak{g}' \to \mathfrak{g}$ is surjective, there is an element $\beta \in \mathfrak{g}^*$

satisfying $i^*(\beta) = \alpha$. In addition, the restriction $i^*\colon \mathcal{O}_{G'}(\beta) \to \mathcal{O}_{G'}(\alpha)$ is also surjective since it is equivariant. Hence, by putting $H = G'$ in Lemma 18, we have

$$\dim \mathcal{O}_G(\beta) \geq \dim \mathcal{O}_{G'}(\beta) \geq \dim \mathcal{O}_{G'}(\alpha) = \dim G'.$$

On the other hand, in the semi-simple case, we already know that the maximum of the dimension of the $\mathrm{Ad}\, G$-orbit in $\mathfrak{g}$ is $\dim G - \operatorname{rank} G$, and the coadjoint representation is equivalent to the adjoint representation. Hence, we have obtained the desired inequality. ∎

From this proof, we know that if the equality in Proposition 17 holds, then there exists a subgroup G' with dimension $\dim G - \operatorname{rank} G$ acting almost freely on an element of $\mathfrak{g}$, which is quite a strong condition on G.

As we stated in Introduction and (3) above, concerning the value rank π itself, we do not have such an upper bound as Proposition 17 for general non-compact real simple Lie groups, and to obtain such an upper bound is a quite interesting and important problem in considering left invariant Poisson structures. In our experience, there actually exist several types of even high dimensional subalgebras of $\mathfrak{g}$ which never admit left invariant symplectic structures, and it is a profound mystery what makes $\mathfrak{g}'$ a subalgebra with (or without) left invariant symplectic structures.

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